Sample Question Sheet 2

No calculators allowed.

1) Logs. Recall that:

If $y = a^x$ then $x = \log_a y$ and the four rules of logarithms are:

i) log(a) + log(b) = log(a X b)
ii) log(a) - log(b) = log(a / b)
iii) c x log(a) = log(a^c)
iv) Change of Base: $\log_a(m) = \log_b(m) \times \log_a(b)$

Note that $\log_{10} x$ may also be written as $\lg(x)$ and $\log_e(x)$ is commonly written as $\ln(x)$.

a) Evaluate the following:

i) $\log_{10} 100$
$\Rightarrow 10^x = 100$
$\therefore x = 2$

ii) $\log_2 (1/8)$
$\Rightarrow 2^x = \frac{1}{8}$
$\therefore x = -3$

iii) $\log_{10} 0.01$
$10^x = 0.01$
$\therefore x = -2$

iv) $\log_2(2\sqrt{2})$
$2^x = 2\sqrt{2} = 2.2^{\frac{1}{2}} = 2^{3/2}$
$\therefore x = 3/2 = 1\frac{1}{2}$

v) $\log_p(p^5) = 5\log_p(p)$
$= 5$

b) If A = lg2 and B = lg3, write the following in terms of A and B.

i) lg(54)
$\lg(54) = \lg(2 \times 3^3)$
$= \lg 2 + \lg 3^3$
$= \lg 2 + 3.\lg 3$
$= A + 3B$

ii) lg(72)
$\lg(72) = \lg(2^3 \times 3^2)$
$= 3\lg 2 + 2\lg 3$
$= 3A + 2B$

iii) $\lg(\frac{9}{16})$
$\lg(9/16) = \lg 9 - \lg 16$
$= \lg 3^2 - \lg 2^4$
$= 2\lg 3 - 4\lg 2$
$= 2B - 4A$

iv) $\lg(\frac{18}{27}) = \lg\left(\frac{2}{3}\right) = \lg 2 - \lg 3$
$= A - B$

b) i) If $6^{x-1} = 3^{x+2}$ find n, m such that $x = \frac{\log_{10} n}{\log_{10} m}$.

$\therefore \log_{10}(6^{x-1}) = \log_{10}(3^{x+2})$
So $(x-1)\log_{10} 6 = (x+2)\log_{10} 3$
$\therefore x\log_{10} 6 - \log_{10} 6 = x\log_{10} 3 + 2\log_{10} 3$
$\therefore x\log_{10} 6 - x\log_{10} 3 = \log_{10} 6 + 2\log_{10} 3$
$\therefore x(\log_{10} 6 - \log_{10} 3) = \log_{10} 6 + \log_{10} 3^2$

ii) Then change base so that x is in $\log_2$.

$\therefore x = \frac{\log_{10} 6 + \log_{10} 9}{\log_{10} 6 - \log_{10} 3} = \frac{\log_{10} 54}{\log_{10}(6/3)} = \frac{\log_{10} 54}{\log_{10} 2}$ So $n = 54$, $m = 2$

Change of base: $\log_{10} 54 = \log_2 54 \times \log_{10} 2$
$\log_{10} 2 = \log_2 2 \times \log_{10} 2$ $\therefore x = \frac{\log_2 54 \times \log_{10} 2}{\log_2 2 \times \log_{10} 2}$
$\therefore x = \log_2 54$

2) Solve the following. (*Hint: Create a new expression for y = ? then substitute it in.*)

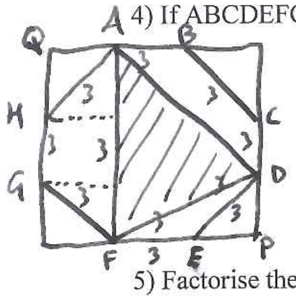
a) $(3^{2x}) - 10(3^x) = -9$ Let $y = 3^x$. So $y^2 = 3^{2x}$.

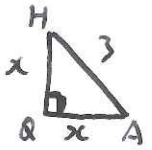
Then $y^2 - 10y + 9 = 0$
$\therefore (y-9)(y-1) = 0 \Rightarrow y = 9$ or $y = 1 \Rightarrow x = 2$ or 0

3) Expand:

a) $(x+y)^3 = (x+y)(x+y)(x+y)$
$= (x^2+2xy+y^2)(x+y) = x^3 + 2x^2y + xy^2 + yx^2 + 2xy^2 + y^3 = x^3 + 3x^2y + 3xy^2 + y^3$

4) If ABCDEFGH is a regular octagon with sides of length 3, what is the area of triangle ADF?





$3^2 = 2x^2$
$\therefore x = \frac{3}{\sqrt{2}}$

So Area AFD $= \frac{1}{2} \times \overline{AF} \times (\overline{FE} + \overline{EP})$
$= \frac{1}{2} \times \left(\frac{3}{\sqrt{2}} + 3 + \frac{3}{\sqrt{2}}\right) \times \left(3 + \frac{3}{\sqrt{2}}\right)$
$= \frac{1}{2}\left(3 + \frac{6}{\sqrt{2}}\right)\left(3 + \frac{3}{\sqrt{2}}\right) = \cancel{\frac{1}{2}(9+36+9\sqrt{2}+18\sqrt{2})}$
$= \frac{1}{2}\left[9 + \frac{9}{\sqrt{2}} + \frac{18}{\sqrt{2}} + \frac{18}{2}\right] = \frac{1}{2}\left(18 + \frac{27}{\sqrt{2}}\right) = 9 + \frac{27\sqrt{2}}{4} = \cancel{\frac{1}{2}(45+27\sqrt{2})}$

5) Factorise the following:

a) $x^3 - x$
$= x(x^2 - 1)$
$= x(x-1)(x+1)$

b) $1 - 2y - 3y^2$
$= (1-3y)(1+y)$

c) $x^4 - 2x^2 - 3$
$= (x^2-3)(x^2+1)$
$= \underbrace{(x-\sqrt{3})(x+\sqrt{3})(x^2+1)}_{\text{optional!}}$

d) $x^2y - 5xy + 4y$
$= y(x-4)(x-1)$

e) $6x^2 - 11x - 7$
$= (3x-7)(2x+1)$

6) Simplify the following:

a) $\frac{x-3}{2x-6} = \frac{(x-3)}{2(x-3)} = \frac{1}{2}$

b) $\frac{2x+3y}{4x^2-9y^2} = \frac{(2x+3y)}{(2x-3y)(2x+3y)} = \frac{1}{2x-3y}$

c) $\frac{x^2+9x+20}{X+4} = \frac{(x+5)(x+4)}{x+4} = x+5$

7) Solve the following inequalities and represent them on a number line: (*use* ∘ *for* < *and* • *for* ≤)

a) $3x+1<5x-2$

$2x>3$

$x>1\frac{1}{2}$

b) $-(x+1)\geq 8$

$(x+1)\leq 8$

$x\leq -9$

c) $3x-7\leq x+4<2x+1$

$2x\leq 11$ $\quad x>3$

$x\leq 5\frac{1}{2}$

d) $x^2>4$

$x>2$ or $x\leq -2$

8) A mother is seven times as old as her daughter. Three years ago she was thirteen times as old. Find the ratio of their ages in 3 years time.

$y=7x$ ①

$(y-3)=13(x-3)=13x-39$

$\therefore y=13x-36$ ②

① - ② ⇒ $0=-6x+36$

$\therefore 6x=36$

$\therefore x=6 \Rightarrow y=42$

$x+3=9,\ y+3=45$ ⇒ In 3 years ratio is 1:5

9) Find the equation of a line that has gradient 7 and passes through (5, 43).

$y=7x+c$

$(5,43) \Rightarrow 43=7\times5+c$

$\therefore c=8$

$\therefore y=7x+8$

10) Find the equation of a line that passes through (5, 8) and (8, 17).

$y=mx+c$

$(5,8) \Rightarrow 8=5m+c$ ①

$(8,17) \Rightarrow 17=8m+c$ ②

② - ① ⇒ $9=3m$

$\therefore m=3$

So $c=-7$

$\therefore y=3x-7$

11) Find the equation of a line parallel to $y=2x+5$ passing through (2, 3).

$y=2x+c$

$(2,3) \Rightarrow 3=2\times2+c$

$\therefore c=-1$

$\therefore y=2x-1$

12) Find the equation of a line perpendicular to $y=2x+1$ passing through (-4, 8). Where do the lines cross? How far is (-4, 8) from the crossing place, and where would it's mirror image be?

$y=-\frac{1}{2}x+c$

$(-4,8) \Rightarrow 8=-\frac{1}{2}\times4+c$

$\therefore c=6$

So $\underline{y=-\frac{1}{2}x+6}$

lines cross: $y=2x+1$ ①

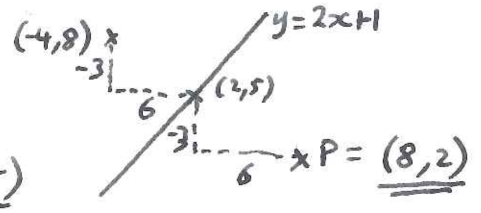
$y=-\frac{1}{2}x+6$ ②

① - ② ⇒ $0=2\frac{1}{2}x-5$

$\therefore x=2$

So $y=5$ $\therefore \underline{(2,5)}$

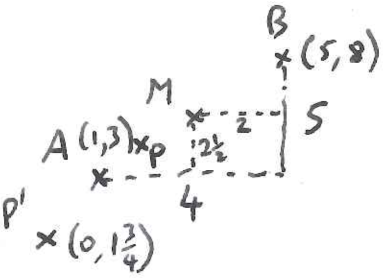
Distance $=\sqrt{(2--4)^2+(5-8)^2}=\sqrt{36+9}=\underline{3\sqrt{5}}$



13) What is the mid-point, M, of A (1, 3) and B (5, 8)? If point P lies on the line AB and divides A and B such that AP:BP = 1:5 what are its coordinates? If you extend the line AB can you find another point P' which satisfies the same ratio?

$M=(3,5\frac{1}{2})$

$P=\left[1+\frac{4}{6},3+\frac{5}{6}\right]=(1\frac{2}{3},3\frac{5}{6})$



Let $P'=(P_x',P_y')$

$5\times(P_x'-1)=(P_x'-5)$ and $5(P_y'-3)=(P_y'-8)$

$\therefore 4P_x'=0$ $\qquad \therefore 4P_y'=7$

$\therefore P_x'=0$ $\qquad \therefore P_y'=1\frac{3}{4}$

So $P_y'=(0,1\frac{3}{4})$